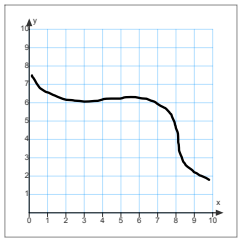
Numerical Integration: The Trapezoid Rule

There are some elementary functions that are difficult to use the limiting process with.
Functions like

$$\sqrt[3]{x}\sqrt{1-x};\quad \sqrt{x}\cos x;\quad \frac{\cos x}{x};\quad \sqrt{1-x^3}$$

One way to approximate a definite integral is to use n trapezoids



We will be using the formula for the area of a trapezoid which is

$$\frac{1}{2}(b_1+b_2)h$$

we will be modifying it to let $b_1 = f(c_i);\ b_2 = f(c_{i-1});\ h = \Delta x$

so then the result is: $\frac{1}{2}(f(c_i)+f(c_{i-\Delta x}))\Delta x$

The Trapezoid Rule

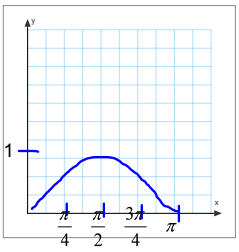
Let f be continuous on [a, b]. The Trapezoid Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]$$

Example:

$\int_0^{\pi} \sin x;\ compare\ n = 4\ and\ n = 8$

Trapezoid Rule for n = 4



You try n=8

Error Analysis

If F has a continuous second derivative on [a, b], then the error E in approximating $\int_a^b f(x)dx$ by the trapezoid rule is

$$E \le \frac{(b-a)^3}{12n^2}[\max|f''(x)|],\ a \le x \le b$$

Determine a value of n such that the Trapezoidal Rule will approximate the value of $\int_0^1 \sqrt{1+x^2}\,dx$ with a value that is less than 0.01.

$$f'(x) = x(1+x^2)^{-1/2} \text{ and } f''(x) = (1+x^2)^{-3/2}$$

Another way to approximate the area under a curve is by Simpson's Rule

$$S = \int_a^b f(x)dx = \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

where [a,b] is partitioned into an even number n of subintervals of equal length h=(b-a)/n.

Example: Use Simpson's Rule with n=4 to approximate $\int_0^2 5x^4 dx$

$$h = (2-0)/4 = 1/2$$

$$S = \frac{1/2}{3}\left(f(0) + 4f(1/2) + 2f(1) + 4f(3/2) + f(2)\right)$$

$$S = 1/6\left(0 + 4(5(\tfrac{1}{2})^4 + 2(5(1)^4 + 4(5(\tfrac{3}{2})^4 + 5(2)^4\right)$$

$$S = 1/6(0 + \frac{20}{16} + 10 + \frac{60}{16} + 80)$$

$$S = 1/6(\frac{80}{16} + 90)$$

$$S = 1/6(5+90) = \frac{95}{6}$$

Error Analysis

If F has a continuous 4th derivative on [a, b], then the error E in approximating $\int_a^b f(x)dx$ by simpson's rule is

$$E \le \frac{(b-a)^5}{180n^4}[\max|f^4(x)|],\ a \le x \le b$$

$\int_0^1 \cos(\pi x)$ less than 0.00001

Assignment

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