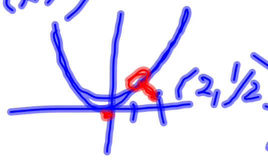
$y=x^2$ $(x, f(x))$

$(2, \frac{1}{2})$



$x^4-4x+\frac{17}{4}$

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$4x^3-4=0$

$4x^3=4$ $(-\infty,1)$ $(1,\infty)$

$\sqrt{(x-2)^2+(x^2-\frac{1}{2})^2}$

$x^3=1$ $f'(x)<0$ $f'(x)>0$

$\sqrt{x^2-4x+4+x^4-x^2+\frac{1}{4}}$

$x=1$ dec   inc

$y=1$

$d=\sqrt{x^4-4x+17/4}$

$(1,1)$

$100=x^4-4x+17/4$

Dec 4-12:23 PM

Differentials

$y=x^2$

$y'=2x\cdot dx$

The derivative of a function can often be used to approximate certain function values with a surprising degree of accuracy. To do this, the concept of the differential of the independent variable and the dependent variable must be introduced.

Nov 29-11:43 AM

The definition of the derivative of a function $y = f(x)$ as you recall is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \frac{\Delta y}{\Delta x}$$

which represents the slope of the tangent line to the curve at some point $(x, f(x))$. If Δx is very small ($\Delta x \neq 0$), then the slope of the tangent is approximately the same as the slope of the secant line through $(x, f(x))$. That is,

$$f'(x) \approx \left[f(x+\Delta x) - f(x) \right] / \Delta x$$

$$\text{or equivalently } f'(x) \cdot \Delta x \approx \underbrace{f(x+\Delta x) - f(x)}_{\Delta y}$$

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The differential of the independent variable x is written dx and is the same as the change in x, Δx. That is,

$x_1 = 7.23$

$x_2 = 7$

$dx = -.23$

$$dx = \Delta x, \Delta x \neq 0$$

$$\text{hence, } f'(x) \cdot dx \approx f(x+\Delta x) - f(x)$$

The differential of the dependent variable y, written dy, is defined to be

$dy = \Delta y = y'$

$$dy = f'(x) \cdot dx \approx f(x+\Delta x) - f(x)$$

$$\text{Because } \Delta y = f(x+\Delta x) - f(x)$$

$$\text{you find that } \boxed{dy = f'(x)\, dx} \approx \Delta y$$

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The conclusion to be drawn from the preceding discussion is that the differential of *y*(*dy*) is approximately equal to the exact change in *y*(Δy), provided that the change in *x* ($\Delta x = dx$) is relatively small. The smaller the change in *x*, the closer *dy* will be to Δy, enabling you to approximate function values close to *f(x)* (Figure 1)

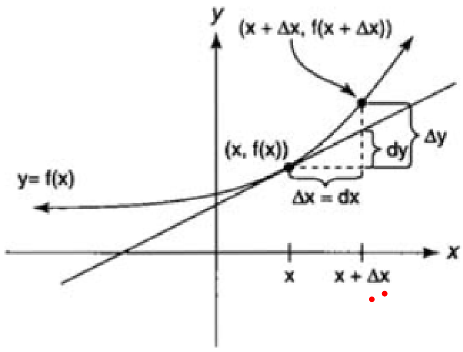

Figure 1
Approximating a function with differentials.

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Example 1: Find *dy* for $y = x^3 + 5x - 1$.

$y = f(x)$

Because $y = f(x) = x^3 + 5x - 1$

$f'(x) = 3x^2 + 5$

$\boxed{dy = f'(x) \cdot dx}$

$dy = (3x^2 + 5) \cdot dx$

$y = 3x^4 - x^2$

$f'(x) = 12x^3 - 2x$

$dy = (12x^3 - 2x)dx$

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Example 2: Use differentials to approximate the change in the area of a square if the length of its side increases from 6 cm to 6.23 cm.

Let x = length of the side of the square. The area may be expressed as a function of x, where $y = x^2$. The differential dy is

$f(x) = x^2$

$f'(x) = 2x$

$$dy = f'(x) \cdot dx$$
$$dy = 2x \cdot dx$$

Because x is increasing from $\boxed{6}$ to 6.23, you find that $\Delta x = dx = .23$ cm; hence,

$$dy = 2\,(6\text{ cm})(.23\text{ cm})$$
$$dy = 2.76\text{ cm}^2$$

The area of the square will increase by approximately 2.76 cm^2 as its side length increases from 6 to 6.23. Note that the exact increase in area (Δy) is 2.8129 cm^2.

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Example 3: Use differentials to approximate the value of $\sqrt[3]{26.55}$ to the nearest thousandth.

Because the function you are applying is $f(x) = \sqrt[3]{x}$, choose a convenient value of x that is a perfect cube and is relatively close to 26.55, namely $x = 27$. The differential dy is

$f(x) = \sqrt[3]{x}$

27 to 26.55

$dx = -.45$

$$dy = f'(x)\,dx$$
$$dy = \frac{1}{3}x^{-2/3}\,dx$$
$$dy = \frac{1}{3x^{2/3}}\,dx$$

Because x is decreasing from 27 to 26.55, you find that $\Delta x = dx = -.45$; hence,

$\sqrt[3]{27^2}$

$\sqrt[3]{(3^3)^2}$

$\sqrt[3]{27 \cdot 27} = 3 \cdot 3 = 9$

$$dy = \frac{1}{3(27)^{2/3}} \cdot (-.45)$$
$$= \frac{1}{27} \cdot -\frac{45}{100}$$
$$dy = -\frac{1}{60}$$

$27^{2/3} = \sqrt[3]{27^2}$

$= \sqrt[3]{27 \cdot 27}$

$\sqrt[3]{27} \cdot \sqrt[3]{27}$

$3 \cdot 3 = 9$

which implies that $\sqrt[3]{26.55}$ will be approximately 1/60 less that $\sqrt[3]{27} = 3$; hence,

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which implies that $\sqrt[3]{26.55}$ will be approximately 1/60 less that $\sqrt[3]{27} = 3$; hence,

$\sqrt[3]{27}$

$\sqrt[3]{26.55} = \sqrt[3]{27} - dy$

$$\sqrt[3]{26.55} \approx 3 - \frac{1}{60}$$
$$\approx 3 - .0167$$
$$\approx 2.9833$$
$$\sqrt[3]{26.55} \approx 2.983$$

to the nearest thousandth.

Note that the calculator value of $\sqrt[3]{27} = 3$ is 2.983239874, which rounds to the same answer to the nearest thousandth!

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Error propagation

The difference between $f(x + \Delta x) - f(x) = \Delta y$

Measurement error is Δx
Exact value is $f(x + \Delta x)$
Measure value is $f(x)$
Propagated error is Δy

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Example:
The radius of a ball bearing is measured to be 0.7 in.. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

$$V = \frac{4}{3}\pi r^3, \ with\ r = 0.7\ and\ -0.01 < \Delta r < 0.01$$

$$\Delta V = dV = 4\pi r^2 dr$$

$$dV = 4\pi(0.7)^2(\pm 0.01)$$

$$\approx \pm 0.06158\ cu.\ in.$$

propagated error is about 0.06

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Relative error and percent of error

Is the error too large or small?

A better answer is given in relative terms which is a comparison of dV and V.

$$\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{\cancel{3}}\pi r^3} = \cdot\frac{3}{\cancel{3}} = \frac{\overset{3}{\cancel{12}}\cancel{\pi}\cancel{r^2}dr}{\cancel{4}\cancel{\pi}r^{\cancel{3}}}$$

$$= \frac{3dr}{r}$$

$$= \frac{3(\pm 0.01)}{0.7}$$

$$\approx \pm 0.0429\ or\ 4.29\%$$

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